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MINIMAX QUADRATIC PROBLEM OF MOTION CORRECTION

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The minimx problem of correction for a linearized model of controllable perturbed motion is considered on the assumption that the available information is restricted to the measurement of the function of the coordinate system [1-6]. Solution of this problem —based on methods of minimax quadratic filtration [7] and optimal control with incomplete data [8] —is brought to a form suitable for computer calculations.

1. Statement of the problem. Let there be given an n -vector controllable system and an m -vector measurement system

$$x^{\cdot}(t) = A(t)x(t) + B(t)u + C(t)v, \quad t_0 - \delta \leqslant t \leqslant \vartheta, \quad \delta > 0 \tag{1.1}$$

$$y(t) = G(t)x(t) + \xi \tag{1.2}$$

where $A(t)$, $B(t)$, $C(t)$ and $G(t)$ are known continuous matrices of related dimensions, and $u = u(t)$ is a q -vector control subject to restriction

$$\langle u'(\cdot)Q(\cdot)u(\cdot)\rangle^{\vartheta} \leqslant \mu^2, \quad Q'(t) = Q(t) > 0 \tag{1.3}$$

$$\left(\langle f(\cdot)\rangle^t = \int_{t_0-\delta}^{t} f(s)\,ds\right)$$

where $Q(t)$ is a continuous and positive definite $q \times q$ matrix, and the prime denotes transposition.

Here and below the symbol $\langle\cdot\rangle^t$ denotes the integral taken over the segment $[t_0 - \delta, t]$ of the related vector or scalar integrand, as is made clear in parentheses in (1.3). When the segment of integration $[t, \tau]$ differs from $[t_0 - \delta, t]$ we write $\langle\cdot\rangle_t^{\tau}$. All considerations in this paper that differ from finite-dimensional are carried out in spaces that are summable with the square of functions which may, possibly, have different metrics. This aspect will not be specifically mentioned below.

It is assumed that the intermediate perturbations $v(\cdot)$, and $\xi(\cdot)$ in systems (1.1) and (1.2) are subject to constraints.

$$I^{\vartheta}(v(\cdot), \xi(\cdot)) = \langle v'(\cdot)R(\cdot)v(\cdot) + \xi'(\cdot)H(\cdot)\xi(\cdot)\rangle^{\vartheta} \leqslant \nu^2 \tag{1.4}$$

$$R'(t) = R(t) > 0, \quad H'(t) = H(t) > \theta$$

Symbols $f_t(\cdot)$ and $f(\cdot \mid t)$ are used when it is necessary to emphasize that function $f = f(\tau)$, $t_0 - \delta \leqslant \tau \leqslant \vartheta$ is considered along segments $[t_0 - \delta, t]$ and $[t, \vartheta]$, respectively.

Definitions. The totality of those and only those vectors $x = x(t)$ which are the ends of trajectories of system (1.1) and which, by virtue of (1.2), generate the required signal $y^*(\tau)$, $t_0 - \delta \leqslant \tau \leqslant t$ by the specified control $u^*(\tau) \equiv 0$, $\tau < t$ under constraints (1.4) on the indetermined perturbations will be called information set $X^*(t) = X(t, y^*(\cdot))$.

Note that, generally speaking, vector $x \in X^*(t)$ may be realized not only in the presence of the unique perturbations $v_t^*(\cdot)$, $\xi_t^*(\cdot)$. The totality of all possible functions $\{v_t^*(\cdot), \xi_t^*(\cdot)\}$, which by virtue of system (1.1) and (1.2) generate signal $y_t^*(\cdot)$ and vector x will be denoted by the symbol $W(y_t^*(\cdot), x)$. Let us determine the quantity

$$\nu^2(y_t^*(\cdot), x) = \min_{v(\cdot),\, \xi(\cdot)} I^t(v(\cdot), \xi(\cdot)), \quad \{v(\cdot), \xi(\cdot)\} \in W(y_t^*(\cdot), x) \tag{1.5}$$

and consider the constraints

$$\langle v'(\cdot)\ R(\cdot)\, v(\cdot) + \xi'(\cdot)\ H(\cdot)\ \xi(\cdot)\rangle_t^\vartheta \leqslant \nu^2 - \nu^2(y_t^*(\cdot), x) \tag{1.6}$$

We introduce in the analysis the set

$$X^\vartheta(u(\cdot \mid t) \mid X^*(t)) = \bigcup \{G(t, x, \vartheta, u(\cdot \mid t)) \mid x \in X^*(t)\} \tag{1.7}$$

where G is the region of attainability [1] of system (1.1) from the state $x = x(t)$ at instant ϑ over all possible $v(\cdot \mid t)$ that are subjected to the constraint (1.6) when $\xi(\cdot \mid t) = 0$ with fixed control $u(\cdot \mid t) \in U(t)$. Here and below $U(t)$ denotes the set of functions $\{u(\cdot \mid t)\}$ subjected to constraint

$$\langle u'(\cdot)\, Q(\cdot)\, u(\cdot)\rangle_t^\vartheta \leqslant \mu^2, \quad t \geqslant t_0 \tag{1.8}$$

Definition 1.2. The totality of those and only those functions $y_\tau(s)$ which coincide with signal $y_t^*(s)$ when $s \leqslant t$, and for $s > t$ are generated by virtue of systems (1.1) and (1.2) (when $u(\cdot \mid t) = 0$) by some quantities x, $v(\cdot \mid t)$, $\xi(\cdot \mid t)$, where $x \in X^*(t)$ and $v(\cdot \mid t)$, $\xi(\cdot \mid t)$ are subject to constraints (1.6), will be called the set of admissible continuations $Y(\tau, y_t^*(\cdot))$ of signal $y_t^*(\cdot)$.

The signal $y_t^*(\cdot)$ realized at instant $t \leqslant \vartheta$ is to be understood as the position of system (1.1) (when $u_t(\cdot) = 0$).

Definition 1.3. We call strategy of correction $U_k = U_k(y_t^*(\cdot))$ the rule by which at each known position of $y_t^*(\cdot)$ only one of the following solutions is acceptable:

a) continue the observation process with zero control in system (1.1) or

b) discontinue observation and pass in system (1.1) to programmed control $u_k(\cdot|t) \in U(t)$ dictated by strategy for the remaining time interval.

Acceptance of the process of solution by strategy U_k begins at instant of time t_0.

Thus, according to Definition 1.3 for each signal $y^*(\cdot) \in Y(\vartheta, y_{t_0}^*(\cdot))$ and the chosen strategy U_k there exists a uniquely defined instant of time $\tau_* = \tau_*(y^*(\cdot), U_k) \in [t_0, \vartheta]$ and the corresponding position of $y_{\tau_*}^*(\cdot)$ in which strategy U_k begins to operate to solution b). The indicated instant τ_* is, obviously, one and the same for all $y(\cdot) \in Y(\vartheta, y_{\tau_*}^*(\cdot))$.

Let there be specified the functional $\Phi(X)$ determined on compact sets $X \subset R^n$ by the condition

$$\Phi(X) = \max\{\varphi(Dx) \mid x \in X\} \tag{1.9}$$

where $\varphi(\cdot)$ is a certain nonnegative convex function on R^d and D is a constant $d \times n$-matrix. We introduce the notation

$$\Psi(u(\cdot \mid t);\ y_t^*(\cdot)) = \Phi(X^\vartheta(u(\cdot \mid t) \mid X^*(t))),\ u(\cdot \mid t) \in U(t) \tag{1.10}$$

$$r(y^*(\cdot), U_k) = \Psi(u_k(\cdot \mid \tau_*);\ y_{\tau_*}^*(\cdot)),\ y^*(\cdot) \in Y(\vartheta, y_{t_0}^*(\cdot)) \tag{1.11}$$
$$\tau_* = \tau_*(y^*(\cdot), U_k)$$

Function $u_k(\cdot \mid \tau_*)$ in formula (1.11) is selected by strategy U_k at instant $t = \tau_*$.

Problem. Find the optimal minimax strategy U_k° for which with any $y^*(\cdot) \in Y(\vartheta, y_{t_0}^*(\cdot))$ the inequalities

$$r(y^*(\cdot), U_k^\circ) \leqslant \Psi(u(\cdot \mid t); y_t^*(\cdot)),\ \forall\, u(\cdot \mid t) \in U(t) \tag{1.12}$$
$$\forall t \in [t_0, \tau_*],\quad \tau_* = \tau_*(y^*(\cdot), U_k^\circ)$$

$$r(y^*(\cdot), U_k^\circ) \leqslant \max_{y_t(\cdot)} \min_{u(\cdot|t)} \Psi(u(\cdot \mid t);\ y_t(\cdot)) \tag{1.13}$$
$$y_t(\cdot) \in Y(t, y_{\tau_*}^*(\cdot)),\ u(\cdot \mid t) \in U(t),\quad t \in [\tau_*, \vartheta]$$

are valid.

If U_k° is the optimal strategy, then any number r° such that

$$r^\circ \geqslant r(y^*(\cdot), U_k^\circ),\quad \forall\, y^*(\cdot) \in Y(\vartheta, y_{t_0}^*(\cdot)) \tag{1.14}$$

is called the assured result of strategy U_k°. It is shown below that an optimal strategy U_k° that entails inequalities (1.12) and (1.13) can be always derived.

Note that condition (1.13) considered for $t = \tau_*$ implies the equality

$$r(y^*(\cdot), U_k^\circ) = \min_{u(\cdot \mid \tau_*)} \Psi(u(\cdot \mid \tau_*);\ y_{\tau_*}^*(\cdot)) \tag{1.15}$$
$$u(\cdot \mid \tau_*) \in U(\tau_*)$$

2. An a posteriori estimate of the state and the programmed control with incomplete data. We shall present an analytical description of the information set $X^*(t)$ on the assumption that system (1.1) $(u(\cdot) = v(\cdot) = 0)$ is entirely observable by signal (1.2) $(\xi(\cdot) = 0)$ along any segment $[t_1, t_2] \subseteq [t_0 - \delta, \vartheta]$. Such assumption is in turn equivalent to complete controllability of the system

$$s^{\cdot} = -sA(t) + \lambda'(t) G(t), \quad t_0 - \delta \leqslant t \leqslant \vartheta \tag{2.1}$$

which is conjugate of (1.1).

We introduce the operator

$$J^t \lambda(\tau) = G(\tau) \langle S(\cdot, \tau) C(\cdot) R^{-1}(\cdot) C'(\cdot) s'(\cdot; \lambda(\cdot))\rangle_\tau^t + H^{-1}(\tau) \lambda(\tau), \; t_0 - \delta \leqslant \tau \leqslant t \tag{2.2}$$

where $S(t, \tau)$ is the fundamental matrix of the conjugate system (2.1) and $s(\tau; \lambda(\cdot))$ is the solution of system (2.1) with the initial condition $s(t_0 - \delta; \lambda(\cdot)) = 0$. It is not difficult to verify that J^t is a linear self-conjugate and strictly positive-definite (coercitive) operator that transforms the space of m-vector functions, which are summable with their square along segment $[t_0 - \delta, t]$, into itself. It follows from the theory of functional analysis [9] that J^t has a bounded inverse operator and, consequently, that the equation $J^t\lambda(\cdot) = \mu(\cdot)$ is uniquely solvable for any right-hand side $\mu(\cdot)$.

We introduce the new scalar product of functions on the assumption that

$$[\lambda(\cdot), \; \mu(\cdot)]^t = \langle \lambda_t'(\cdot) J^t \mu_t(\cdot) \rangle^t \tag{2.3}$$

Let $f^*(t, \cdot)$ and $F(t, \cdot)$ be solutions of equations

$$J^t f^*(t, \cdot) = y_t^*(\cdot), \quad J^t F(t, \cdot) = G(\cdot) S(t, \cdot) \tag{2.4}$$

where $y_t^*(\cdot)$ is the position obtaining at instant $t \geqslant t_0$. In the second equality of (2.4) $F(t, \cdot)$ is an $m \times n$-matrix function whose column $f^i(t, \cdot)$ are solutions of equations

$$J^t f^i(t, \cdot) = g^i(t, \cdot), \quad i = 1, \ldots, n$$

where $g^i(t, \cdot)$ are columns of matrix $G(\cdot) S(t, \cdot)$.

The following statement is valid [7].

Lemma 2.1. The inclusion $x \in X(t, y^*(\cdot))$ is equivalent to the inequality

$$(x - x_0(t))' P(t) (x - x_0(t)) \leqslant \nu^2 - h^2(t) \tag{2.5}$$

$$P(t) = [F'(t, \cdot), F(t, \cdot)]^t$$
$$h^2(t) = [f^*(t, \cdot), f^*(t, \cdot)]^t - x_0'(t) P(t) x_0(t) \tag{2.6}$$

where $x_0(t)$ is a vector determined by formula

$$x_0(t) = P^{-1}(t) d(t), \; d'(t) = [f^*(t, \cdot), F(t, \cdot)]^t = s(t; f^*(t, \cdot)) \tag{2.7}$$

The confirmation of Lemma 2.1 arises from the following results.

Lemma 2.2 Of all functions $v_t(\cdot)$, $\xi_t(\cdot)$ which satisfy (almost everywhere on $[t_0-\delta, t]$) the identity

$$-G(\tau)\langle S(\cdot,\tau)\,C(\cdot)\,v(\cdot)\rangle_\tau{}^t+\xi(\tau)=J^t\,b(t,\tau),\quad t_0-\delta\leqslant\tau\leqslant t$$

where $b(t,\cdot)=f^*(t,\cdot)-F(t,\cdot)\,x,$ are functions of the form

$$v_t{}^1(\tau)=-R^{-1}(\tau)\;C'(\tau)\;s'(\tau;\;b(t,\cdot)),\qquad \xi_t{}^1(\tau)=H^{-1}(\tau)\;b(t,\tau)$$

that yield the minimum of formula (1.5) equal to $[b(t,\cdot),b(t,\cdot)]^t$.

Proof. Let us consider the identity with respect to $\lambda_t(\cdot)$

$$\langle(v'(\cdot)R(\cdot)+s(\cdot;\;\lambda_t(\cdot))C(\cdot))R^{-1}(\cdot)(R(\cdot)v(\cdot)+C'(\cdot)s'(\cdot;\lambda_t(\cdot)))\rangle^t+\langle(\xi'(\cdot)H(\cdot)-\lambda_t'(\cdot))H^{-1}(\cdot)(H(\cdot)\xi(\cdot)-\lambda_t(\cdot))\rangle^t\equiv I^t(v(\cdot),\;\xi(\cdot))-2[\lambda_t(\cdot),\;b(t,\cdot)]^t+[\lambda_t(\cdot),\;\lambda_t(\cdot)]^t\geqslant 0$$

Setting here $\lambda_t(\cdot)=b(t,\cdot)$, we obtain the inequality

$$I^t(v(\cdot),\xi(\cdot))\geqslant[b(t,\cdot),b(t,\cdot)]^t$$

which, obviously, transforms functions $v_t{}^1(\cdot)$, $\xi_t{}^1(\cdot)$ into equalities. The lemma is established.

Lemma 2.2 and formulas (2.6) and (2.7) imply the equality

$$\nu^2(y_t{}^*(\cdot),\;x)=h^2(t)+(x-x_0(t))'\;P(t)\,(x-x_0(t)) \tag{2.8}$$

Note that owing to the complete observability of system (1.1) by signal (1.2), matrix $P(t)$ (see (2.6)) is nondegenerate and the information set $X^*(t)$ is thus a nondegenerate n-dimensional ellipsoid with its center at $x_0(t)$. The quantities $F(t,\tau)$, $f^*(t,\tau)$ (2.4) $P(t)$, $h^2(t)$ (2.6) and $d(t)$, $x_0(t)$ (2.7) are differentiable functions of time t [7] and satisfy the following differential equations:

$$\frac{\partial F(t,\tau)}{\partial t}=-F(t,\tau)(A(t)+C(t)R^{-1}(t)C'(t)P(t)),\quad t\geqslant\tau \tag{2.9}$$

$$F(\tau,\tau)=H(\tau)\,G(\tau) \tag{2.10}$$

$$P^{\boldsymbol{\cdot}}(t)=-A'(t)\,P(t)-P(t)\,A(t)-P(t)\,C(t)\,R^{-1}(t)\,C'(t)\times P(t)+G'(t)\,H(t)\,G(t),\;P(t_0-\delta)=0$$

$$d^{\boldsymbol{\cdot}}(t)=-(A'(t)+P(t)\;C(t)\;R^{-1}(t)\;C'(t))\,d(t)+G'(t)\;H(t)\,y^*(t),\quad d(t_0-\delta)=0 \tag{2.11}$$

$$\frac{\partial f^*(t,\tau)}{\partial t}=-F(t,\tau)C(t)R^{-1}(t)C'(t)d(t),\qquad t\geqslant\tau \tag{2.12}$$

$$f^*(\tau,\tau)=H(\tau)\;y^*(\tau)$$

$$\frac{dh^2(t)}{dt}=f_1{}^{*\prime}(t)\,H^{-1}(t)\,f_1{}^*(t),\quad h^2(t_0-\delta)=0 \tag{2.13}$$

$$f_1^* (t) = H (t) (y^* (t) - G (t) x_0 (t))$$

$$x_0^{\cdot} (t) = A (t) x_0 (t) + P^{-1} (t) G' (t) f_1^* (t), \quad t \geqslant t_0 \tag{2.14}$$

Equations (2.10), (2.11), (2.13) and (2.14) completely define the dynamics of change of the information set $X^* (t)$ (2.5).

It is thus possible to note that the operator equations (2.4) play only a subsidiary part. The calculation of parameters of ellipsoid (2.5) can be effected by direct integration of Eqs. (2.10) and (2.11) on the basis of incoming information $y^* (\tau)$, $t_0 - \delta \leqslant \tau \leqslant t$. Functions $x_0 (t)$ and $h^2 (t)$ are determined by formulas

$$x_0 (t) = P^{-1} (t) \; d (t), \quad h^2(t) = \langle f_1^{*\prime} (\cdot) H^{-1} (\cdot) \; f_1^* (\cdot)\rangle^t \tag{2.15}$$

We pass to the solution of the problem of guidance programming, in other words of finding a control $u^\circ (\tau)$, $t_0 \leqslant t \leqslant \tau \leqslant \vartheta$ such that

$$\min \{\Psi (u (\cdot); \; y_t^* (\cdot)) \mid u (\cdot) \in U (t)\} = \Phi^\circ (y_t^* (\cdot)) \tag{2.16}$$

From formulas (1.6), (1.7), (1.9), (1.10), (2.5) and (2.8) we have the equality

$$\begin{aligned} \Phi^\circ (y_t^* (\cdot)) = {} & \min_{u(\cdot)\in U(t)} \max \{s (t; l) \, x_0 (t) + \langle s (\cdot; \; l) \, B (\cdot) \, u (\cdot)\rangle_t^\vartheta + \\ & s (t; \; l) \, x + (\nu^2 - h^2 \, (t) - x' \, P \, (t) \, x)^{1/2} \, (\alpha \, (t; \; l))^{1/2} - \\ & \varphi^* \, (l) \mid x' \, P \, (t) x \leqslant \nu^2 - h^2 \, (t), \;\; l \in R^n\} \\ \alpha \, (t; \; l) = {} & \langle s \, (\cdot; \; l) \, C \, (\cdot) \, R^{-1} \, (\cdot) \, C' \, (\cdot) \, s' \, (\cdot; \; l)\rangle_t^\vartheta \\ s \, (t; \; l) = {} & l'DS \, (t, \; \vartheta) \end{aligned} \tag{2.17}$$

where $\varphi^* (\cdot)$ is a convex function conjugate of $\varphi (\cdot)$ [10]. To determine in (2.17) the intrinsic maximum with respect to x we use the relationship

$$\max \, \{s'x + (\beta^2 - x' \, Px)^{1/2} \, r \mid x' \, Px \leqslant \beta^2\} = \mid \beta \mid (r^2 + s'P^{-1}s)^{1/2}$$
$$r \geqslant 0, \; P' = P > 0, \; x \in R^n \tag{2.18}$$

The maximum is attained here on the element $x^\circ = |\beta| \, (r^2 + s'P^{-1}s)^{-1/2} P^{-1}s$. Using (2.18) and transposing in (2.17) the minimum with respect to $u (\cdot)$ and maximum with respect to l, we finally obtain

$$\begin{aligned} \Phi^\circ (y_t^* (\cdot)) = {} & \max_l \, \{s \, (t; \; l) \, x_0 \, (t) - \mu \, (\beta \, (t; \; l))^{1/2} + \\ & (\operatorname{conc} g) \, (t; \; l)\}, \;\; l \in R^d \end{aligned} \tag{2.19}$$

$$\begin{aligned} \beta \, (t; \; l) = {} & \langle s \, (\cdot; \; l) \, B \, (\cdot) \, Q^{-1} \, (\cdot) \, B' \, (\cdot) \, s' \, (\cdot; \; l)\rangle_t^\vartheta \\ g \, (t; \; l) = {} & (\nu^2 - h^2 \, (t))^{1/2} \, (\alpha \, (t; \; l) + \\ & s \, (t; \; l) \, P^{-1} \, (t) \, s' \, (t; \; l))^{1/2} - \varphi^* \, (l) \end{aligned} \tag{2.20}$$

where the symbol $(\operatorname{conc} g) \, (t; \; l)\}$ denotes the upper envelope of function

$g(t; l)$ i.e. the smallest closed concave function which majorizes $g(t; l)$. The following statement is valid.

Theorem 2.1 The optimal control $u^\circ(\cdot)$ in problem (2.16) always exists and satisfies the principle of minimum

$$\langle s(\cdot;\ l^\circ)\ B(\cdot)\ u^\circ(\cdot)\rangle_t^\vartheta = \min\ \{\langle s(\cdot;\ l^\circ)\ B(\cdot)\ u(\cdot)\rangle_t^\vartheta \mid u(\cdot) \in U(t)\} \tag{2.21}$$

where l° is the extreme element in problem (2.19).

Using condition (2.21) we obtain the optimum control

$$u^\circ(\tau) \equiv -\mu Q^{-1}(\tau)\ B'(\tau)\ s'(\tau;\ l^\circ)\ (\beta(t;\ l^\circ))^{-1/2} \tag{2.22}$$

if $\beta(t;\ l^\circ) \neq 0$.

One of the difficulties of solving problem (2.16) is the determination of the quantity $(\mathrm{conc}\ g)(t;\ l)$. Only in certain cases is it possible to obtain the upper envelope of function $g(t;\ l)$ in explicit form. Let, for instance, function $\varphi(\cdot)$ be an Euclidean norm. It is then possible to show that

$$(\mathrm{conc}\ g)(t;\ l) = (\nu^2 - h^2(t))^{1/2}\ (\pi_0^2(t)\ (1 - l'l) + l'\ P_1(t)\ l)^{1/2},\quad l'l \leqslant 1$$

where $\pi_0^2(t)$ is the highest eigenvalue of matrix

$$P_1(t) = \langle DS(\cdot,\ \vartheta)\ C(\cdot)\ R^{-1}(\cdot)\ C'(\cdot)\ S'(\cdot,\ \vartheta)\ D'\rangle_t^\vartheta + DS(t,\ \vartheta)\ P^{-1}(t)\ S(t,\ \vartheta)\ D'$$

In that case formula (2.19) assumes the form

$$r^*(t) = \Phi^\circ(y_t^*(\cdot)) = \max_{l'l \leqslant 1}\ \{s(t;\ l)\ x_0(t) - \mu\ (\beta(t;\ l))^{1/2} + (\nu^2 - h^2(t))^{1/2}\ (\pi_0^2(t)\ (1 - l'l) + l'\ P_1(t) l)^{1/2}\} \tag{2.23}$$

3. **Solution of the problem of correction.** Let us revert to the problem formulated in Sect. 1. We assume that function $\varphi(\cdot)$ (see (1.9)) is an Euclidean norm.

Let at instant $t \geqslant t_0$ position $y_t^*(\cdot)$ be realized. In that position it is possible to obtain, in spite of the assured result $r^*(t)$ (2.23), one more number

$$r^*(\tau,\ t) = \max\ \{\Phi^\circ(y_\tau(\cdot)) \mid y_\tau(\cdot) \in Y(\tau,\ y_t^*(\cdot))\},\quad \tau \geqslant t \tag{3.1}$$

which defines the prediction of the assured result of control on the basis of information obtained by instant t. Calculating, first, $r^*(\tau,\ t)$ for all $\tau \in [t,\ \vartheta]$, we can finally obtain

$$r_*^\circ(t) = \min\ \{r^*(\tau,\ t) \mid t \leqslant \tau \leqslant \vartheta\} \tag{3.2}$$

and compare this quantity with $r^*(t)$.

We form the equation

$$r^* (t) - r_*^\circ (t) = 0, \quad t \geqslant t_0 \tag{3.3}$$

We call the strategy of correction U_k^e extremal, if as the instant $\tau_* = \tau_* (y^* (\cdot), U_k^e)$ of observation termination for each of signals $y^* (\cdot) \in Y (\vartheta, y_{t_0}^* (\cdot))$ we take the smallest root of Eq. (3.3), and select function (2.22) as the control $u_k (\cdot | \tau_*)$ imposed by strategy on segment $[\tau_*, \vartheta]$. The following statement is valid.

Theorem 3.1. The extremal strategy U_k^e is optimal and assures the result

$$r (y^* (\cdot), U_k^e) = \Phi^\circ (y_{\tau_*}^* (\cdot)) = r^* (\tau_*) = r_*^\circ (\tau_*) \leqslant r^\circ = r_*^\circ (t_0), \quad \forall\ y^* (\cdot) \in Y (\vartheta, y_{t_0}^* (\cdot)) \tag{3.4}$$

Proof. The validity of the theorem follows from the definition of strategy U_k^e, formulas (3.1) and (3.2) and the inequality $\Phi^\circ (y_t^* (\cdot)) > r^* (\tau_*)$, $\forall t \in [t_0, \tau_*)$. This inequality follows in turn from that the assumption

$$\min \{\Phi^\circ (y_t^* (\cdot)) \mid t \in [t_0, \tau_*)\} = r^* (t_1) \leqslant r^* (\tau_*)$$

where $t_1 < \tau_*$, contradicts condition $r^* (t_1) > r_*^\circ (t_1)$ which is satisfied in conformity with the definition of strategy U_k^e. The assumption that $r^* (\tau_*) > r_*^\circ (t_0)$ also leads to a contradiction, since then at some instant $t^1 \in [t_0, \vartheta]$ we have $r^* (\tau_*) > r_*^\circ (t_0) = r^* (t^1, t_0)$, which is impossible.

Note that the estimate $r_*^\circ (t_0)$ assured by strategy U_k^e cannot be improved in the sense that for any instant $t^1 \in [t_0, \vartheta]$ signal $y_t^* (\cdot) \in Y (t, y_{t_0}^* (\cdot))$ for which

$$r^* (t) = \Phi^\circ (y_t^* (\cdot)) = r^* (t, t_0) \geqslant r_*^\circ (t_0)$$

can be obtained.

The last inequality means that, having observed signal $y^* (\tau)$ up to instant t, no selection of programed control $u (\cdot | t) \in U (t)$ in the problem considered, can yield a better result than $r_*^\circ (t_0) < r^* (t_0)$.

We thus find that the instant $\tau_* = \tau_* (y^* (\cdot), U_k^e)$ of observation termination is determined by continuous observation of signal $y^* (t)$, calculation of parameters of the information set $X^* (t)$, and also by the continuous computation of numbers $r^* (t)$ (2.23) and $r_*^\circ (t)$ (3.2). However the assured results $r_*^\circ (t_0)$ may be always derived by simpler procedures.

In fact, if $r_*^\circ (t_0) = r^* (t_0)$, it is no longer necessary to continue observation. If, however, $r_*^\circ (t_0) < r^* (t_0)$, it is possible to carry out at once observation up to the instant

$$t^1 = \max \{t \mid r^* (t, t_0) = r_*^\circ (t_0)\} > t_0 \tag{3.5}$$

Then along segment $[t_0, t^1]$ a signal which may not be the worst for the observer may be realized, and we can repeat the prediction procedure by comparing the numbers $r_*^\circ (t^1)$ and $r^* (t^1) \leqslant r_*^\circ (t)$, etc. until equality (3.3) is achieved. In the latter case further observation is unnecessary, and the programmed control (2.22) is to be applied on segment $[t, \vartheta]$

It should be emphasized the described procedure for the determination of solution is, first of all, aimed at obtaining the assured result $r_*^\circ(t_0)$ while the extremal strategy U_k^e makes it possible to exploit to the highest degree the unsuccessful from the adversary's point of view choice of signal $y^*(\cdot) \in Y(\vartheta, y_{t_0}^*(\cdot))$ and obtain the lowest possible value of the quantity

$$\Phi^\circ(y^*_{\tau_*}(\cdot)) = \Psi(u^\circ(\cdot \mid \tau_*); y^*_{\tau_*}(\cdot))$$

To make the reasoning in Sect. 3 completely strict we shall show the attainability of maximum in (3.1) and calculate that maximum. We revert to Eqs. (2.9)-(2.14) which define the dynamics of variation of set $X^*(t)$. The initial conditions $x_0(t_0) = P^{-1}(t_0)\, d(t_0)$, $h^2(t_0)$ for Eqs. (2.13) and (2.14) are obtained from observation of signal (1.2) on the initial segment $[t_0 - \delta, t_0]$ It will be readily seen that the evolution of these quantities is uniquely determined by specifying function $f_1^*(t)$. Let us consider the set of all functions $\{f_1(\cdot)\}$ specified on segment $[t, \tau]$, $t \geqslant t_0$ and subjected to the constraint

$$\langle f_1'(\cdot) H^{-1}(\cdot)\, f_1(\cdot)\rangle_t^\tau \leqslant \nu^2 - h^2(t) \tag{3.6}$$

where $h^2(t)$ is determined by formulas (2.13) and (2.15).

Lemma 3.1. Signal $y_\tau(\cdot)$ is an admissible continuation of signal $y_t^*(\cdot)$ (i.e. $y_\tau(\cdot) \in Y(\tau, y_t^*(\cdot))$), if and only if there exists function $f_1(\cdot)$ that satisfies the inequality (3.6) and such that

$$y_\tau(\alpha) = G(\alpha)\, x_0(\alpha) + H^{-1}(\alpha)\, f_1(\alpha), \quad t \leqslant \alpha \leqslant \tau \tag{3.7}$$

where $x_0(\alpha)$ is the solution of the equation

$$x_0^{\cdot}(\alpha) = A(\alpha)\, x_0(\alpha) + P^{-1}(\alpha)\, G'(\alpha)\, f_1(\alpha), \quad t \leqslant \alpha \leqslant \tau \tag{3.8}$$

with known initial conditions $x_0(t) = P^{-1}(t)\, d(t)$.

Proof. From (3.7) and (3.8) we have

$$y_\tau(\alpha) = G(\alpha)\, S(\tau, \alpha)\, x_0(\tau) - \langle G(\alpha)\, S(\cdot, \alpha)\, P^{-1}(\cdot)\, G'(\cdot)\, f_1(\cdot)\rangle_\alpha^\tau + H^{-1}(\alpha)\, f_1(\alpha),\ t_0 - \delta < \alpha \leqslant \tau \tag{3.9}$$

and when $\alpha \leqslant t$ function $y_\tau(\alpha)$ is the same as the obtained signal $y^*(\alpha)$. Using Lemma 2.2 and formulas (2.8)-(2.14) we conclude from this that signal (3.9) and vector $x_0(\tau)$ can be actually obtained in the system (1.1), (1.2), for instance, in the presence of perturbations $v_t^1(\cdot)$, $\xi_t^1(\cdot)$ indicated in Lemma 2.2 where $t = \tau$ and function $b(\tau, \cdot)$ are the solution of equation

$$J^\tau b(\tau, \cdot) = y_\tau(\cdot) - G(\cdot)\, S(\tau, \cdot)\, x_0(\tau)$$

Hence, if the continuation of signal $y_t^*(\cdot)$ is specified by formulas (3.7) and (3.8), such continuation is admissible. The converse assertion of the lemma is evident.

It follows from Lemma 3.1 that there exists a one-to-one correspondence between signal $y_\tau(\cdot) \in Y(\tau, y_t^*(\cdot))$ and functions $f_1(\alpha), t \leqslant \alpha \leqslant \tau$. Taking this and (2.23) into account, we rewrite formula (3.1) in the form

$$r^*(\tau, t) = \max_{f_1(\cdot)} \max_{l'l \leqslant 1} \{s(t; l)\, x_0(t) + \langle s(\cdot; l) \times P^{-1}(\cdot)\, G'(\cdot)\, f_1(\cdot)\rangle_t^\tau - \mu\, (\beta(\tau; l))^{1/2} + (\nu^2 - h^2(t) - \langle f_1'(\cdot)\, H^{-1}(\cdot)\, f_1(\cdot)\rangle_t^\tau)^{1/2} (\pi_0^2(\tau)\, (1 - l'l) + l'\, P_1(\tau)\, l)^{1/2}\} \tag{3.10}$$

Using a formula analogous to the finite dimensional equality (2.19) we obtain

$$r^*(\tau, t) = \max_{l'l \leqslant 1} \{s(t; l)\, x_0(t) - \mu\, (\beta(\tau; l))^{1/2} + (\nu^2 - h^2(t))^{1/2} (\pi_0^2(\tau)\, (1 - l'l) + l'P_1(\tau)\, l + \langle s(\cdot; l)\, P^{-1}(\cdot) \times G'(\cdot)\, H(\cdot)\, G(\cdot)\, P^{-1}(\cdot)\, s'(\cdot; l)\rangle_t^\tau)^{1/2}\} \tag{3.11}$$

The expression in (3.10) has obviously a maximum with respect to $f_1(\cdot)$, which shows that maximum is attainable in (3.1). Formula (3.11) shows that function $r^*(\tau, t)$ is continuous with respect to variables τ, t $(t_0 \leqslant t \leqslant \tau \leqslant \vartheta)$.

Thus the procedure of acceptance of solution of extremal strategy U_k^e reduces to the following sequence of operations which must be continuous in time.

1°. Determination of parameters of ellipsoid $X^*(t)$, i.e. of quantities $P^{-1}(t)$, $x_0(t)$ and $h^2(t)$, by solving the related differential equations (2.10), (2.11) and (2.13).

2°. Calculation of the quantities $r^*(t)$, $r^*(\tau, t)$ and $r_*^\circ(t)$ by formulas (2.23), (3.11) and (3.2) respectively.

3°. Determination of the minimum root of Eq. (3.3) by comparing $r^*(t)$ and $r_*^0(t)$.

In practice the described procedure is realized on a computer in the form of a discrete scheme with a small time step.

4. Example. Let us consider a one-dimensional system whose state $x(t)$ is measured under conditions of some interference

$$\dot{x} = u + v, \quad -\delta \leqslant t \leqslant \vartheta, \quad \delta > 0 \tag{4.1}$$

$$y(t) = x(t) + \xi \tag{4.2}$$

Perturbations $v(\cdot), \xi(\cdot)$ and the control $u(\cdot)$ are constructed by (4.3) and condition (4.4) respectively,

$$\int_{-\delta}^{\vartheta} (v^2(s) + \xi^2(s))\, ds \leqslant \nu^2 \tag{4.3}$$

$$\int_{-\delta}^{\vartheta} u^2(s)\, ds \leqslant \mu^2 \tag{4.4}$$

In this example Eqs. (2.9)-(2.14) assume the form

$$p^{\cdot}(t) = 1 - p^2, \quad p(-\delta) = 0 \tag{4.5}$$
$$d^{\cdot}(t) = -p(t)\, d(t) + y^*(t), \quad d(-\delta) = 0$$
$$dh^2(t)/dt = f_1^{*2}(t), \quad f_1^*(t) = y^*(t) - x_0(t), \quad h^2(-\delta) = 0$$
$$x_0^{\cdot}(t) = p^{-1}(t)\, f_1^*(t), \quad x_0(t) = p^{-1}(t)\, d(t)$$

From this we find that $p(t) = \mathrm{th}\,(t+\delta)$. In this case formulas (2.23) and (3.11) are of the form

$$r^*(t) = \Phi^\circ(y_t^*(\cdot)) = \max\{|x_0(t)| - \mu(\vartheta - t)^{1/2},\ 0\} + (\nu^2 - h_2(t))^{1/2}(\vartheta - t + p^{-1}(t))^{1/2} \tag{4.6}$$
$$r^*(\tau, t) = \max_{|l| \leqslant 1} \times$$
$$\left\{ l x_0(t) - \mu |l| (\vartheta - \tau)^{1/2} + (\nu^2 - h^2(t))^{1/2} \left(\vartheta - \tau + p^{-1}(\tau) + l^2 \int_t^\tau p^{-2}(s)\, ds \right)^{1/2} \right\}$$

Let us use the following constants: $\mu = \nu = 1,\ \vartheta = 3$ and $\delta = 1$, and specify the simulated signal as $y^*(s) \equiv 1,\ -1 \leqslant s \leqslant 3$. It follows then from Eqs. (4.5) that $d(t) = \mathrm{th}\,(t+1),\ x_0(t) \equiv 1$ and $h^2(t) \equiv 0$. Formulas (4.6) now assume the form

$$r^*(t) = \begin{cases} \lambda(t), & t \leqslant 2 \\ 1 - (3-t)^{1/2} + \lambda(t), & t > 2 \end{cases}$$
$$r^*(\tau, t) = \begin{cases} \lambda(\tau), & t \leqslant \tau \leqslant 2 \\ 1 - (3-\tau)^{1/2} + (1 + \mathrm{cth}\, 3)^{1/2}, & t = 2 \end{cases}$$
$$\lambda(t) = (3 - t + \mathrm{cth}\,(t+1))^{1/2}$$

Since function $\lambda(t)$ monotonically decreases, hence it immediately follows from the obtained formulas that the maximum root of Eq. (3.3) is two. Maximum with respect to l in the second of formulas (4.6) obtains when $t = \tau = 2$ on element $l = 1$. Hence the optimal programmed control is $u^\circ(s \mid 2) \equiv -1$. (see (2.22)).

Thus, if at instant $t = 2$ we pass to the control $u^\circ(s \mid 2) \equiv -1$, at the final instant $\vartheta = 3$ we obtain the functional

$$\Psi(u^\circ(\cdot \mid 2);\, y_2^*(\cdot)) = (1 - \mathrm{cth}\, 3)^{1/2}$$

If, however, observation is continued up to instant $\tau > 2$ (on condition that $y_2^*(s) \equiv 1$), then even with the best selection of programed control $u^\circ(\cdot \mid \tau)$ the functional $\Psi(u^\circ(\cdot \mid \tau);\, y_\tau^*(\cdot))$ may reach the value $(1 + \mathrm{cth}\, 3)^{1/2} + 1 - (3-\tau)^{1/2} > (1 + \mathrm{cth}\, 3)^{1/2}$.

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